Analysis of a Tensegrity-Based Parallel Platform Device

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Abstract

This paper presents a new parallel platform device that is based on tensegrity. Tensegrity structures are comprised of struts that are in compression and ties that are in tension. A 3-3 tensegrity structure has three struts and nine ties. In this paper a platform device is created by replacing the three struts with three variable length legs. Also, three of the ties are replaced by a combination of three springs in series with three adjustable length non-compliant ties. Lastly, the remaining six ties form two triangles and these ties are replaced by rigid bodies to yield a platform-like device where the top platform is connected to the base by six leg connectors, i.e. the three variable length legs and the three spring and variable length tie combinations. A reverse position and energy analysis is conducted to determine the lengths of the three variable length legs and the three variable length ties in order to position and orient the top platform as desired at a specified potential energy value. It is shown that the top platform can be moved to any arbitrary position if the screw coordinates of an externally applied wrench are specified.

Introduction

Buckminster Fuller assigned a meaning to the word tensegrity. The word refers to the phenomena that all objects in the universe exert a pull on each other and thus the universe is tensionally continuous. The universe is also compressionally discontinuous [1]. A tensegrity structure is an illustration of this phenomenon on a man-sized scale.

A tensegrity structure is pictured in Figure 1. This structure is a triangular tensegrity prism. A tensegrity structure consists of multiple members, some of which are solely in tension (ties) while the others are in compression (struts). None of the compression members come in contact with any other compression member. However, the members in tension are different in this respect.

Any vertex of the structure can be connected to another point on the structure by tracing a line along tensional members. This is evidence of the structure’s tensional continuity. The members labeled “S” in Figure 1 are compressional. The remaining members are ties, and thus can only be in tension. This structure, without the application of an external force, can only be in equilibrium at two positions where one is simply a mirror image of the other reflected through the base plane.

![Figure 1: Tensegrity Structure](image)

A parallel prism is the root of a tensegrity prism. The parallel prism has ties that are parallel and have fixed lengths. A tensegrity prism is created by taking a parallel prism and rotating the top plane about its central, perpendicular axis by an angle \( \alpha \), and then
inserting non-compliant members connecting the erstwhile end-points of the diagonals of the planes made of the parallel ties [2]. For a triangular parallel prism there are three such planes, thus there are three non-compliant members in a triangular tensegrity prism.

Tensegrity prisms, like parallel prisms, can be of any degree polygon. Kenner [3] found the rotation angle, $\alpha$, for the general tensegrity prism as

$$\alpha = \frac{\pi}{n} \frac{\pi}{n}$$  \hspace{1cm} (1)

where $n$ is the number of sides in the polygon of the upper or lower plane.

A different property of tensegrity prisms is calculated to be independent of the value of $n$. Knight et al. [2] show that, due to the arrangement of its members; a triangular tensegrity structure has instantaneous mobility. This is a characteristic of all tensegrity prisms.

Tensegrity prisms share another interesting characteristic in that the insertion of compliant ties causes them to be deployable from a bundled position. Duffy et al. [4] analyze the deployable characteristics of elastic tensegrity prisms, as do Tibert [5] and Stern [6]. Tensegrity prisms have this attribute because the potential energy stored in their elastic members is greater when bundled than when in the equilibrium position of Figure 1. Indeed, this figure illustrates a position of minimum potential energy. The allure of tensegrity structures encompasses more than their characteristics of motion.

Burkhart [7] fabricates and analyzes domes using triangular tensegrity prisms. The triangular tensegrity prisms that he uses have smaller tops than bottoms. Others have studied the uses of tensegrity prisms when the top and bottom are the same size, but the lengths of the side ties are variable.

Tran [8] discusses a device comprised of three compliant ties and three non-compliant struts, all of which have adjustable lengths. Each compliant leg consists of a non-elastic cable in parallel with an elastic member, a spring. These legs can be called side ties. This device is a triangular tensegrity prism with variable lengths. Oppenheim [9] also deals with adjustable-member-length tensegrity prisms. The aim of these devices is to allow a tensegrity structure to achieve varied positions.

These devices can have different stiffness values for identical postures because of the variable nature of the compliant member lengths. Skelton [10] discusses using tensegrity structures with variable member lengths in wing construction. Doing this would allow the wing to have adjustable stiffness or shape. The thesis presented here evaluates a method of calculating the conditions necessary to attain variable configurations of a triangular-tensegrity-prism based device.

This paper will focus on the analysis of a parallel platform mechanism that incorporates compliance and tensegrity. The objective of the work is to show how the position and orientation of the top platform can be controlled together with the stiffness of the device at that pose.

**Tensegrity Based Platform**

Figure 2 depicts the mechanism that is considered in this study. There is similarity with the tensegrity structure shown in Figure 1. Here the top three ties and the bottom three ties which each form a triangle have been replaced by rigid bodies that are labeled the top platform and base platform. The three struts are replaced by variable length struts that are connected to the base by ball joints and to the top by Hooke joints. The three side ties have been replaced by a combination of three springs in series with three adjustable length non-compliant ties. As with the struts, the side ties are connected to the base and top platforms by ball and Hooke joints respectively. In Figure 1, the struts and side ties of the tensegrity structure meet at a point at the top and bottom. Here, the connection points of the struts and side ties have

![Figure 2: Tensegrity-Based Parallel Platform Device](image-url)
been displaced from one another to simplify the assembly of the device.

**Problem Statement**

The problem to be addressed in this paper is to determine the lengths of the three struts and the three non-compliant ties that will position and orient the top platform as desired with a specified total potential energy stored in the three springs. Specifically the problem statement is presented as follows (see Figure 3:

- given:
  - the lengths of the sides of the top and bottom platforms of the device ($l_t$, $l_b$),
  - the ratio of the separation distance of adjacent connection points to the length of the top or bottom platform ($\sigma_t$, $\sigma_b$),
  - the position and orientation of the top platform relative to the bottom, defined by a $4 \times 4$ transformation matrix ($T$),
  - the spring free lengths ($l_{04}$, $l_{05}$, $l_{06}$),
  - the spring constants ($k_4$, $k_5$, $k_6$),
  - the potential energy stored in the springs ($U$),
  - the screw along which an external wrench acts ($S_{ext}$).

- find:
  - the length of each strut ($L_1$, $L_2$, $L_3$)
  - the length of cable in each compliant leg ($l_{c4}$, $l_{c5}$, $l_{c6}$)
  - the spring deformations ($\delta_4$, $\delta_5$, $\delta_6$)

Note that subscripts 1, 2, and 3 refer to the three adjustable length struts while subscripts 4, 5, and 6 refer to the side ties which are comprised of a spring and adjustable length tie in series.

**Determination of Plücker Line Coordinates**

Based on the given information it should be apparent that the coordinates of all the connection points of the struts and side ties, i.e. the center of the ball and Hooke joints, can all be calculated in terms of a coordinate system attached to the base platform. These calculations are not shown here.

The next step of the solution is to determine the Plücker coordinates of the lines along the struts and side ties. The Plücker coordinates of a line are defined by a vector along the direction of the line together with the moment of the line with respect to the origin. These coordinates are homogeneous in that multiplying the two vectors by some scalar $\lambda$ defines the same infinite line in space. For this analysis, unique coordinates will be written by defining the direction of the line by the unit vector $S_i$.

Thus the Plücker coordinates of the line along leg connector $i$ will be written as

$$S_{li} = \begin{bmatrix} S_i & S_{0li} \end{bmatrix}$$  \hspace{1cm} (2)

where as stated, $S_i$ is a unit vector that defines the direction of the line and $S_{0li}$ is the moment of the line with respect to the origin of the reference coordinate system. This moment is calculated as

$$S_{0li} = r_i \times S_i$$  \hspace{1cm} (3)

where $r_i$ is any point on the line. In this analysis the direction of the unit vector for each of the struts will be from the bottom platform towards the top platform. For the three side ties, the direction of the unit vector will be from the top platform towards the bottom platform.

Since the coordinates of the center points of all the ball and Hooke joints are known in terms of a reference system attached to the base platform, the Plücker line coordinates of the lines along each of the six known with respect to this same reference system.
Definition of a Screw and a Wrench

A screw was defined in the definitive work of Ball [11] as a line with a pitch. Pitch is defined as “the rectilinear distance through which the nut is translated parallel to the axis of the screw while the nut is rotated through the angular unit of circular measure.” The coordinates of a screw with pitch \( h \) may be written as

\[
\begin{bmatrix}
    S_1 \\
    S_2 \\
    \vdots \\
    S_n \\
\end{bmatrix}
\]

where

\[
S_{0i} = S_{0Li} + h S_i.
\]

The screw coordinates are homogeneous in that multiplying \( S_i \) by any scalar \( \lambda \) defines the same screw. In this analysis, unique screw coordinates will be defined by requiring that the vector \( S \) be a unit vector.

It was also shown by Ball that a pure force acting on a rigid body can be modeled by multiplying the coordinates of the line of action, \( S_{0Li} \), by the scalar magnitude of the force. It was also shown that a pure moment acting on a rigid body could be modeled by a force acting on a line at infinity. The coordinates of a pure moment reduce to

\[
\begin{bmatrix}
    m \\
    S \\
\end{bmatrix}
\]

where \( m \) is the magnitude of the moment and \( S \) is a unit vector that is perpendicular to the plane that the moment acts in.

By using this representation for forces and moments, Ball showed that the resultant of a set of forces and moments acting on a rigid body can be calculated as the sum of the individual coordinates as

\[
f_{\text{res}} S_{\text{res}} = f_1 S_{11} + f_2 S_{12} + \ldots + m_{n-1} S_{n-1} + m_n S_n.
\]

In general, the resultant will not be a pure force or a pure moment, but rather will be a force magnitude acting upon a screw with pitch \( h \). The geometric interpretation of this result is that of a pure force acting on the line of action of the screw together with a moment acting along the direction of the screw. This representation of a force magnitude times a screw was defined by Ball as a wrench.

Determination of Spring Elongation Values

The required elongation of each of the springs will be determined from a force balance equation acting on the top platform combined with the total potential energy equation for the mechanism. There are seven wrenches acting on the top platform. Six are pure forces acting along the lines of the struts and side ties and the seventh is an external wrench whose unitized screw coordinates were given in the problem statement. In order for the device to be in equilibrium, the sum of these wrenches must be equal to zero. The force and moment balance equation for the top platform can be written as

\[
f_1 S_{11} + f_2 S_{12} + f_3 S_{13} + f_4 S_{14} + f_5 S_{15} + f_6 S_{16} + f_{\text{ext}} S_{\text{ext}} = 0
\]

The subscripts 1, 2, and 3 refer to the three struts while the subscripts 4, 5, and 6 refer to the three side ties. The directions of the lines were previously defined such that positive force values for the parameters \( f_i \) will mean that the struts are in compression and the side ties are in tension. It should be noted that all the screw coordinates of (8) are known with the scalar force values \( f_i \) being unknown.

Equation (8) can be rearranged as

\[
f_1 S_{11} + f_2 S_{12} + f_3 S_{13} + f_5 S_{15} + f_6 S_{16} + f_{\text{ext}} S_{\text{ext}} = -f_4 S_{14}
\]

and this equation can be written in matrix format as

\[
\begin{bmatrix}
    f_1 \\
    f_2 \\
    f_3 \\
    f_5 \\
    f_6 \\
    f_{\text{ext}} \\
\end{bmatrix}
= -f_4 S_{14}
\]

where the term \([S_{11} \ S_{12} \ S_{13} \ S_{15} \ S_{16} \ S_{\text{ext}}]\) is a 6×6 matrix whose columns are the unitized coordinates of the lines along legs 1, 2, 3, 5, and 6 and the unitized screw coordinates of the given external wrench.

Both sides of (10) are now divided by the force magnitude \( f_4 \). The result can be written as

\[
\begin{bmatrix}
    f_1' \\
    f_2' \\
    f_3' \\
    f_5' \\
    f_6' \\
    f_{\text{ext}}' \\
\end{bmatrix}
= -S_{14}
\]

where

\[
f_n' = \frac{f_n}{f_4}.
\]
The ratios of the six forces to \( f_4 \) are found by multiplying both sides of (11) by the inverse of the matrix containing the screw information. This can be written as

\[
\begin{bmatrix}
  f_1' \\
  f_2' \\
  f_3' \\
  f_5' \\
  f_6' \\
  f_{\text{ext}}'
\end{bmatrix}
= \left[\begin{bmatrix}
  S_{L,1} & S_{L,2} & S_{L,3} & S_{L,5} & S_{L,6} & S_{L,\text{ext}}
\end{bmatrix}\right]^T \mathbf{s}_{L,4} \tag{13}
\]

Thus the ratios of the force magnitudes to \( f_4 \) are known.

The equation for the total amount of potential energy stored in the three legs \( (U) \) is

\[
U = \frac{k_4 \delta_4^2 + k_5 \delta_5^2 + k_6 \delta_6^2}{2} \tag{14}
\]

where \( \delta_i \) is the deformation of the spring along leg \( i \). The force in each spring is related to the spring constant and elongation as

\[
f_i = k_i \cdot \delta_i \tag{15}
\]

Substituting (15) into (14) yields

\[
U = \frac{\left( \frac{f_4^2 + f_5^2 + f_6^2}{k_4 + k_5 + k_6} \right)^2}{2} \tag{16}
\]

Substituting \( f_5 = f_4 f_5' \) and \( f_6 = f_4 f_6' \) yields

\[
U = \frac{\left( \frac{f_4^2 + f_5^2 + f_6^2}{k_4 + k_5 + k_6} \right)^2}{2} \tag{17}
\]

This equation can be solved for \( f_4 \) as

\[
f_4 = \sqrt{\frac{2U \cdot k_4 k_5 k_6}{k_4 + k_5 + k_6}} \frac{f_5 f_6'}{f_4^2} \tag{18}
\]

The forces in all the legs and the force magnitude applied to the external screw can now be determined as

\[
f_i = f_4 \cdot f_i' \tag{19}
\]

Lastly the elongations of the three springs can be determined as

\[
\delta_i = \frac{f_i}{k_i}, i = 4, 5, 6 \tag{20}
\]

**Determination of Strut Lengths and Cable Lengths**

Determination of the strut lengths is trivial since the coordinates of the end points of the strut are known in terms of the reference coordinate system attached to the base of the manipulator. The cable lengths can now be determined since the spring elongation values have been calculated.

The total distance between the end points of the side ties, \( L_4, L_5, \) and \( L_6 \) can be calculated in the same manner as the length of the struts since the end point coordinates are known. As shown in Figure 3, this total length equals the length of cable that is deployed plus the free length of the spring and the elongation of the spring. Thus the length of cable to be deployed can be calculated as

\[
l_{oi} = L_i - \delta_i - l_{ci}, i = 4, 5, 6 \tag{21}
\]

**Numerical Example 1**

In example one the top and bottom platforms are of equal size. The joints meet at three points per platform. The three spring constants are equal, as are their free lengths. There is no pitch to the screw of action of the external wrench. That screw passes through the center of each platform. The coordinate system attached to the base is defined as having its origin at the center of the ball joint for leg 1. Its \( x \) axis passes through center of the ball joint for leg 2. Its \( z \) axis is perpendicular to the plane defined by the ball joint points for legs 1, 2, and 3. The coordinate system attached to the top platform is defined as having its origin at the center of the Hooke joint for leg 4. Its \( x \) axis passes through center of the Hooke joint for leg 5. Its \( z \) axis is perpendicular to the plane defined by the Hooke joint points for legs 4, 5, and 6. These coordinate systems will be used in all the example problems.

The following data was specified:

\[
\begin{align*}
  l_i &= l_6 = 20.0 \text{ cm}, \\
  \sigma &= 0.0, \\
  k_4 &= k_5 = k_6 = 20.0 \text{ N/cm}, \\
  l_{04} &= l_{05} = l_{06} = 3.0 \text{ cm}, \\
  U &= 40.0 \text{ N cm},
\end{align*}
\]

\[
\begin{aligned}
  b^T \mathbf{t} &= \begin{bmatrix}
    0.893 & 0.325 & 0.312 & 8 \\
    -0.326 & 0.944 & -0.051 & 5 \\
    -0.312 & -0.056 & 0.949 & 18 \\
    0 & 0 & 0 & 1
  \end{bmatrix},
\end{aligned}
\]
The units of the first three rows of the fourth column of $^B_T$ are cm while the other elements of the matrix are dimensionless. The last three elements of $S_{ext}$ have units of cm.

The lengths of the three struts were computed as the distance between their end points as

$L_1 = 20.322$ cm,
$L_2 = 13.231$ cm,
$L_4 = 18.759$ cm.

The Plücker coordinates of the lines along the struts and side ties were computed to be

$S_{l_1} = \begin{bmatrix} 0.394 \\ 0.246 \\ 0.886 \\ 0 \\ 0 \\ 16.214 \\ -4.504 \end{bmatrix}$, $S_{l_2} = \begin{bmatrix} 0.442 \\ -0.114 \\ 0.889 \\ 0 \\ -17.790 \\ 4.313 \\ -2.287 \end{bmatrix}$, $S_{l_3} = \begin{bmatrix} 0.669 \\ 0.041 \\ 0.742 \\ 12.85 \\ -7.419 \\ 0 \end{bmatrix}$,

$S_{l_4} = \begin{bmatrix} 0.540 \\ -0.225 \\ -0.811 \\ 0 \\ 16.966 \end{bmatrix}$, $S_{l_5} = \begin{bmatrix} -0.581 \\ 0.690 \\ -0.431 \\ -7.470 \\ 4.313 \end{bmatrix}$, $S_{l_6} = \begin{bmatrix} -0.703 \\ -0.564 \\ -0.434 \\ 0 \end{bmatrix}$

Using (13) to solve for the ratio of the force magnitudes to $f_4$ yields

$\begin{bmatrix} f_1' \\ f_2' \\ f_3' \\ f_4' \\ f_5' \\ f_6' \\ f_{ext}' \end{bmatrix} = \begin{bmatrix} 1.583 \\ 1.821 \\ 1.91 \\ 1.41 \\ 1.386 \\ -2.846 \end{bmatrix}$.

The remaining force magnitudes are determined by multiplying each calculated $f_i'$ by $f_4$ to yield

$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_{ext} \end{bmatrix} = \begin{bmatrix} 28.73 \\ 33.046 \\ 34.664 \\ 25.584 \\ 24.822 \\ -51.649 \end{bmatrix}$ N.

From (20) and (21), the length of each non-compliant cable is determined to be

$l_{c_1} = 18.297$ cm ,
$l_{c_2} = 23.007$ cm ,
$l_{c_6} = 27.850$ cm.

**Numerical Example 2**

In this example the top and bottom platforms are of unequal size. The end joints are fixed at six points per platform as shown in Figure 2. The three spring constants are also unequal. Their free lengths are the same. The screw of action of the external wrench has a pitch of 0.3 cm. The following information was specified:

$l_t = 18.0$ cm ,
$l_b = 22.0$ cm,
$\sigma_t = 0.07$ ,
$\sigma_b = 0.16$ ,
$k_4 = 18.0$ N/cm ,
$k_5 = 23.0$ N/cm ,
$k_6 = 30.0$ N/cm ,
$l_{a_0} = l_{b_5} = l_{b_6} = 3.0$ cm ,

$U = 150.0$ N cm ,

$^{B}_T = \begin{bmatrix} 0.669 & -0.743 & 0 & 5 \\ 0.732 & 0.659 & -0.174 & -6 \\ 0.129 & 0.116 & 0.985 & 16 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

$S_{ext} = \begin{bmatrix} -0.389 \\ -0.167 \\ 0.906 \\ 8.203 \\ -11.309 \\ 1.773 \end{bmatrix}$.
The lengths of the three struts were computed as the distance between their end points as 
\[
L_1 = 17.527 \text{ cm},
\]
\[
L_2 = 20.062 \text{ cm},
\]
\[
L_4 = 23.16 \text{ cm}.
\]
The Plücker coordinates of the lines along the struts and side ties were computed to be 
\[
\begin{bmatrix}
0.263 \\
-0.275 \\
0.925 \\
0 \\
0 \\
13.585 \\
5.094
\end{bmatrix}
= \begin{bmatrix}
-0.289 \\
0.312 \\
0.905 \\
0 \\
-19.914 \\
11.248 \\
6.855
\end{bmatrix},
\]
\[
\begin{bmatrix}
15.056 \\
-9.094 \\
0.817 \\
15.571 \\
-8.990 \\
4.484
\end{bmatrix}.
\]

Using (13) to solve for the ratio of the force magnitudes to \( f_4 \) yields 
\[
\begin{bmatrix}
f_{11}' \\
f_{12}' \\
f_{13}' \\
f_{14}' \\
f_{15}' \\
f_{16}' \\
f_{\text{ext}}'
\end{bmatrix} = \begin{bmatrix}
-0.234 \\
-0.711 \\
1.998 \\
0.19 \\
0.558 \\
4.309
\end{bmatrix}.
\]

Next, \( f_4 \) is calculated from the energy equation, (18) as 
\[
f_4 = 66.757 \text{ N}.
\]
The remaining force magnitudes are determined by multiplying each calculated \( f_i' \) by \( f_4 \) to yield 
\[
\begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5 \\
f_6 \\
f_{\text{ext}}
\end{bmatrix} = \begin{bmatrix}
-15.596 \\
-47.435 \\
133.406 \\
12.673 \\
36.92 \\
287.642
\end{bmatrix} \text{ N}.
\]

From (20) and (21), the length of each non-compliant cable is determined to be 
\[
l_{c_4} = 15.056 \text{ cm},
\]
\[
l_{c_5} = 17.235 \text{ cm},
\]
\[
l_{c_6} = 16.417 \text{ cm}.
\]

Conclusion

This paper has introduced a new parallel platform device that incorporates tensegrity. It is shown that the top platform can be moved to an arbitrary position and orientation relative to the base and simultaneously attain a desired potential energy for the system. A significant discovery was the fact that the screw upon which the equilibrium wrench is applied (which is required except for the case of pure tensegrity where the line coordinates of the six legs become linearly dependent) is arbitrary.

Future work will focus on the incorporation of additional leg connectors which in effect will remove the requirement that an external wrench be applied to the top platform to maintain equilibrium. Also work will focus on the incorporation of varying spring constants and introducing damping into the leg connectors so that the vibrational properties of the top platform can be controlled.

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